A SUPPLEMENT TO A THEOREM OF MERKER AND PORTEN: A SHORT PROOF OF HARTOGS' EXTENSION THEOREM FOR $(n-1)$-COMPLETE COMPLEX SPACES

MIHNEA COLŢOIU

1. Introduction

The well-known Hartogs' extension theorem states that for every open subset $D \subset \mathbb{C}^n$, $n \geq 2$, every compact subset $K \subset D$ such that $D \setminus K$ is connected the holomorphic functions on $D \setminus K$ extend to holomorphic functions on D. For a simple and short $\bar{\partial}$ proof see [E]. A long and very involved proof of this result on 33 pp. ,using Morse theory, in the spirit of Hartogs' original idea [Ha] of moving discs to get the extension,was recently obtained by J. Merker and E. Porten [M-P1].

The Hartogs' theorem was generalized to $(n-1)$-complete manifolds (in the sense of A. Andreotti and H. Grauert [A-G]) by A. Andreotti and D. Hill [A-H] using cohomological results ($\bar{\partial}$ method). In their forthcoming paper [M-P2] J. Merker and E. Porten observed that in the singular case "it is at present advisable to look for methods avoiding $\bar{\partial}$ methods, because such tools are not yet available" and "the essence of the present article is to transfer such an approach to $(n-1)$-complete general complex spaces, where the $\bar{\partial}$ techiques are still lacking, with some new difficulties due to singularities". J. Ruppenthal [R] developped a $\bar{\partial}$ machinery for proving the Hartogs' extension theorem on Stein spaces with isolated singularities.

The main result of J. Merker and E. Porten [M-P2], which generalizes Andreotti-Hill theorem [A-H] for singular spaces, can be stated as follows:

Theorem 1.1. *Let X be a normal $(n-1)$-complete space ($n = dimX$), $D \subset\subset X$ a relatively compact open subset, $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then every holomorphic function on $D \setminus K$ can be extended to a holomorphic function on D.*

In fact they proved this result even for the extension of meromorphic functions (previously considered in the smooth case by V. Koziarz and F. Sarkis [K-S]) but we shall consider in this short note only the holomorphic extension. The 20 pages proof of Merker and Porten [M-P2] is also based on their previous paper [M-P1] on 33 pp., so putting together one gets about 50 pages which are very technical.

We will give in this short note a 1 page proof for Theorem 1.1., using the $\bar{\partial}$ method on the resolution of singularities, especially the Takegoshi relative vanishing theorem [T], (

Submitted on November 12^{th} 2008.

see also [O]), which gives even a more general statement valid on cohomologically $(n-1)$-complete spaces (and without the assumption that D is relatively compact). Namely one has:

Theorem 1.2. *Let X be a n-dimensional normal cohomologically $(n-1)$-complete complex space, $D \subset X$ an open subset, $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then every holomorphic function on $D \setminus K$ can be extended to a holomorphic function on D.*

The ideas of the proof are essentially contained in the paper [C-S]

2. Proof of the result

For the basic definitions of q-convex functions, q-complete complex space we reffer to [A-G]. We also recall that a complex space X is called cohomologically q-complete if one has the vanishing of the cohomology groups $H^i(X, \mathcal{F}) = 0$ for every $i \geq q$ and every $\mathcal{F} \in Coh(X)$. By the main result of [A-G] a q-complete space is cohomologically q-complete (a counter-example to the converse is still unknown). For a complex manifold X we denote by K_X its canonical sheaf (associated to the canonical line bundle). Let X be a complex (reduced) space and $\pi : \tilde{X} \to X$ a resolution of singularities (which exists by [A-H-V], [B-M]). The following result , due to K. Takegoshi [T] (see also T. Ohsawa [O]), will be fundamental for our proof:

Theorem 2.1. *Let $\pi : \tilde{X} \to X$ be a resolution of singularities of a complex space X. Then one has the following vanishing for the higher direct images: $R^i \pi_* K_{\tilde{X}} = 0$ if $i \geq 1$*

Let us also recall that by Grauert's coherence theorem [G] $\pi_* K_{\tilde{X}}$ is a coherent sheaf on X. If moreover X is assumed to be cohomologically $(n-1)$-complete it then follows that $H^i(X, \pi_* K_{\tilde{X}}) = 0$ if $i \geq n-1$. By Theorem 2.1. the maps $H^i(X, \pi_* K_{\tilde{X}}) \to H^i(\tilde{X}, K_{\tilde{X}})$ are isomorphisms, so that one gets the vanishing of the cohomology group $H^i(\tilde{X}, K_{\tilde{X}}) = 0$ if $i \geq n-1$. By Serre duality [S] one gets the vanishing of the first cohomology group with compact supports $H^1_c(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Moreover the arguments of L. Ehrenpreis [E] (see also [Ho]) show without any modification that the following holds: If $\tilde{X}$ is a complex connected non-compact manifold such that $H^1_c(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ then for every open subset $\tilde{D} \subset \tilde{X}$ and for every compact subset $\tilde{K} \subset \tilde{D}$, such that $\tilde{D} \setminus \tilde{K}$ is connected, it follows that every holomorphic function on $\tilde{D} \setminus \tilde{K}$ can be extended to a holomorphic function on $\tilde{D}$. Applying this result to $\tilde{D} = \pi^{-1}(D)$ and $\tilde{K} = \pi^{-1}(K)$, where $\pi : \tilde{X} \to X$ is a resolution of singularities for X, one gets immediately Theorem 1.2., since $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ (by the normality of X).

References

[A-G] A. Andreotti and H. Grauert: *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193-259.

[A-H] A. Andreotti and C.D. Hill: *E.E. Levi convexity and the Hans Lewy problem I and II*, Ann. Sc. Norm. Sup. Pisa 26 (1972), 325-363, 747-806.

[A-H-V] J.M. Aroca, H. Hironaka, J.L. Vincente : *Desingularization theorems*, Mem. Math. Jorge Juan, No. 30, Madrid, 1977.

[B-M] E. Bierstone and P. Milman: *Canonical desingularisation in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128(1997), no. 2, 207-302.

[C-S] M. Coltoiu and A. Silva : *Behnke-Stein theorem on complex spaces with singularities.* Nagoya J. Math. 137 (1995), 183-194.

[E] L. Ehrenpreis: *A new proof and an extension of Hartogs' extension theorem*, Bull. Amer. Math. Soc. 67 (1961), 507-509.

[G] H. Grauert: *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Structuren*, Inst. Hautes Études Sci. Publ. Math. no. 5 , 1960, 64 pp.

[Ha] F. Hartogs : *Zur theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, unbesondere über die Darstellung derselben durch Reichen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann. 62 (1906), no. 1, 1-88.

[Ho] L. Hörmander : *An introduction to complex analysis in several variables*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto,1966, 208 pp.

[K-S] V. Koziarz and F. Sarkis: *Problème du bord dand les variétés q-convexes et phénomèn de Hartogs-Bochner*, Math. Ann. 321 (2001), no.3, 569-585.

[M-P1] J. Merker and E. Porten: *A Morse theoretical proof of the Hartogs extension theorem*, J. Geom. Anal. 17 (2007), no.3, 513-546.

[M-P2] J. Marker and E. Porten: *The Hartogs' extension theorem on* $(n-1)$*-complete complex spaces*, preprint, arXiv:0704.3216 (to appear in J. reine angew. Math).

[O] T. Ohsawa : *A vanishing theorem for proper direct images*, Publ. RIMS 23(1987),no. 2, 243-250.

[R] J. Ruppenthal : *A* $\bar{\partial}$ *theoretical proof of Hartogs' extension theorem on Stein spaces with isolated singularities*, J. Geom. Anal. 18 (2008),no.4

[S] J. P. Serre : *Un théorème de dualité*, Comment. Math. Helv. 29(1955),9-26.

[T] K. Takegoshi: *Relative vanishing theorems in analytic spaces*, Duke Math. J. 52 (1985), no. 1, 273-279.

M. Colţoiu: Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, RO-014700, Bucureşti, Romania.

E-mail address: Mihnea.Coltoiu@imar.ro